COEFFICIENT BOUNDS FOR THE FUNCTION IN THE CLASS OF MODIFIED JONQUI

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ABSTRACT

In this work, by considering a general subclass of univalent functions we define a new class $S(\alpha, \Psi i_s)$ associated with the Jonquiere's function (popularly known as polylogarithm function). Subordination principle was employed to obtain the upper bounds for the first few coefficients of the class defined. Furthermore, non-linear functional $|a_3 \mu a_2^2|$ (the classical Fekete-Szego inequality) for the function in the class was established. Conclusively, consequences of certain choices of parameters involved in the results were pointed out. The results further established geometric properties of the Jonquiere's function associated with univalent functions.

Keywords: Jonquiere's function subordination, Fekete-Szego inequality, analytic coefficient bounds, polylogarithm

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Introduction

Let U be the unit disk $\{z \in C : |z| < 1\}$, and let A be the class of all analytic functions in U, which are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1.1)$$

where C is the Complex plane (or the z-plane). Equation (1.1) is the normalized form of $f \in A$, satisfying the conditions $f(0) = 0$ and $f'(0)=1$.

We further denote by K the subclass of S consisting of convex functions. So that $f \in K$ if and only if for

$$z \in U, \qquad Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0.$$

Meanwhile, let denote by Mα, the class of analytic functions in U with $f(0) = 0$, $f'(0)=1$. and which are α - convex in U.

$$M_\alpha := \{f \in s \text{ and } Re\, J(a, f; z) > 0,\ z \in u\}, \quad (1.2)$$

where

$$J(a, f; z) = (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right). \quad (1.3)$$

We remark th $M_1 = K$ The class denoted by M_a and defined by (1.2) is known as the class of a - convex functions (see, (Georgia, 2005)).

The Jonquiére's function

The Jonquiere's function (popularly known as Polylogarithm function) is defined by a power series in s:

$$Li_s(z) = \sum_{n}^{\infty} \frac{Z^n}{n^s} \left(|Z| < 1; s \in C\right). \qquad (1.4)$$

The definition is valid for arbitrary Complex order *s* and for all Complex arguments *z* with $|z| < 1$; it can be extended to $|z| \geq 1$ by the process of analytic continuation. In other words, it is defined in the Complex plane over the open unit disk. It is also denoted by $f(z, s)$ and equal to $Li_s(Z) = \phi(z, s, 1)$, where (z, s, a) is the Lerch transcendent.

Remarks:

From (1.4), the following are obvious:

(i) $Li_0(z) = z + s\,{}_{n=2}^{8} z^n = \frac{z}{1- z}$, for $s = 0$.

It is well known that $Li_0(z) \in K$ which implies that $Li_0(z) \in A$. It is fairly well known that the geometric series $Li_0(z) = \frac{z}{1 \ z}$ acts as the identity element under convolution: that $f * Li_0(z) = f$, (see (Duren, 1983)). $Li_0(z)$ is the recursive definition of $Li_s(z)$.

(ii) $Li_0(0) = 0$ and $Li_0'(0) = 1$ is the condition for normalizing $Li_0(z)$. Thus $Li_0(z) \in A$ of the form (1.1).

The Jonquiére's function appears in several fields of mathematics and in many physical problems. It has attracted a great deal of attention over the last two centuries or so and so has a long history connected with some of the great mathematicians of the past. Historically, the classical polylogarithm function was invented in 1696 by Leibniz and Bernoulli according to (Gerhardt and Leibniz, 1971). Also, information in respect of details studies on polylogarithm function are readily available for interested reader. These include (Cvijović 2007) and (Jodrá 2008).

Recent works on Jonquiére's (polylogarithm) function and its generalized forms can be found in (Alhindin and Darus, 2014); (Altinkaya &Yalcin, 2016); (Akgül, 2017); (Akgül, 2018); (Abdul Rahaman *et al.,* 2018); (Sofo, 2019); (Oladipo, 2019); (Olukoya and Oyekan, 2020) and (Oyekan and Awolere, 2020) to mention but a few. Most of them being problems of estimating coefficient $|\alpha_n|$ for $n \geq 2$ is still an open problem.

Method of Estimation (Preliminaries)

For a better acquaintance with the content of this paper we shall recall some definitions that are relevant to our main result as will be used in subsequent section. In addition, useful lemma that is essential for deriving one of our main results will also be stated

Definition 2.1:

Let P denote the class consisting of analytic functions p(z) in u with positive real part such that $p(0) = 1$, $z \in u$.

This class of functions is known as Carathéodory and has the form:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 \ldots = 1 + \sum_{n=1}^{\infty} p_n z^n, \qquad (2.1)$$

which are analytic in *U*.(see Duren1983)).

Definition 2.2:

Let $\Psi i_s(z)$ be a function of the form (1.4) for all $s \in N$:

$$\Psi i_s(z) = \frac{1}{z} Li_s(z) = \frac{1}{z}\left(\sum_{n=1}^{\infty} \frac{z^n}{n^s}\right) = 1 + \frac{z}{2^s} + \frac{z^2}{3^s} + \frac{z^3}{4^s} \ldots, for\ s \in N. \qquad (2.2)$$

We say that $\Psi I_s(z)$ defined by (2.2) is the modified Jonquiére's (or Polylogarithm) function.

Note that every $s \in N \subset C$ implies $\Psi i_s(z) \subset Li_s(z)$.

Remark:

The function $\Psi i_s(z)$ is a special case of Jonquiére's (polylogarithm) function which obviously belongs to class P. In other words, $Re\ (\Psi i_s(z)) > 0$.

Definition 2.3:

An analytic function *f* is said to be subordinate to another analytic function *g*, written as $f(z) \prec g(z)$ $(z \in U)$ if there exists a Schwarz function w, which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in U, then we have the following equivalence:

$f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$, (see (Pommerenke, 1975)).

Definition 2.4: A function $f \in A$ is said to be in the class $S(\alpha, \Psi i_s)$, $0 = \alpha = 1$, if the following subordination holds:

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \Psi i_s(z). \quad (2.3)$$

Motivated by the work of (Altinkaya and Yalcin, 2016) in which the authors considered a subclass of univalent functions and obtained coefficients expansion using Chebyshev polynomials; we in turn hereunder, obtained coefficient bounds for a class of univalent function using the Special case of Jonquiere's (polylogarithm) function given by (2)

Lemma 1.

Let $\omega(z) \in \Omega$, then for any complex number μ

$$|c_2 - \mu c_1^2| = \max\{1, |\mu|\}. \quad (2.4)$$

This result is sharp for $\omega(z) = z^2$ or $\omega(z) = z$ (see, (Keogh & Merkes, 1969)).

MAIN RESULTS

In this section we shall state and prove the coefficient bounds $|a_2| - |a_4|$ for $f(z) \in S(\alpha, \Psi i_s)$.

Coefficient bounds for the function class $S(\alpha, \Psi i_s)$

Theorem 3.1: Let the function $f(z)$ given by (1.1) be in the class $S(\alpha, \Psi i_s)$. Then

$$|a_2| = \frac{1}{\alpha+1}$$

$$|a_3| = \frac{1}{2.2^s(2\alpha+1)} + \frac{3}{2(2\alpha+1)(\alpha+1)} + \frac{1}{2(2\alpha+1)} - \frac{1}{(2\alpha+1)(\alpha+1)^2}$$

$$|\alpha_4| = \frac{3\alpha+1}{3.3^s} + \frac{1}{2^s}\left\{\left(\frac{6\alpha+2}{3\alpha+1}\right) + \left(\frac{3\alpha+1}{\alpha+1}\right) + \left[\frac{6\alpha-2}{3(2\alpha+1)}\right] - \left[\frac{3\alpha^2-29}{6(2\alpha+1)(\alpha+1)}\right]\right\}$$

$$+ \frac{1}{3(\alpha+1)^2}\left\{(6\alpha+2) + \left[\frac{6\alpha^2+44\alpha-7}{2(2\alpha+1)(\alpha+1)}\right] - \left[\frac{9\alpha^2+90\alpha+29}{2(2\alpha+1)}\right]\right\}$$

$$+ \frac{1}{3(2\alpha+1)}\left\{(6\alpha^2+9\alpha+2) - \left(\frac{3\alpha^2+14\alpha+5}{\alpha+1}\right)\right\} + \frac{8\alpha+5}{3(\alpha+1)}$$

Proof: Let $f \in S(\alpha, \Psi i_s)$. Then from (2.2) and (2.3), we have

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \le 1 + \frac{1}{2^s}\omega(z) + \frac{1}{3^s}\omega^2(z) + \frac{1}{4^s}\omega^3(z) + \dots \quad (3.1)$$

for some analytic function $\omega(z)$ such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U$,

$$|\omega(z)| = |c_1 z + c_2 z^2 + c_3 z^3 + \dots| < 1, z \in U, \quad (3.2)$$

where

$$(c_j) \le 1, \quad \forall\ j \in N; \quad (3.3)$$

Therefore

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \le 1 + \frac{c_1}{2^s}z + \frac{c_2}{3^s}z^2 + \left(\frac{c_3}{2^s} + \frac{c_1^3}{3^s}\right)z^3 + \left(\frac{c_4}{2^s} + \frac{3c_1^2 c_2}{3^s}\right)z^4 \dots \quad (3.4)$$

By simplifying equation (3.4) and equating coefficients, we have

$$a_2(\alpha+1) = c_1 \Rightarrow a_2 = \frac{c_1}{\alpha+1} \quad (3.5)$$

$$2a_3(2\alpha+1) = c_2 + 2^{-s}c_1^2 + 3a_2c_1 - 2a_2^2 \quad (3.6)$$

Using (3.5) in (3.6), we have

$$2a_3(2\alpha+1) = c_2 + 2^{-s}c_1^2 + 3c_1\left(\frac{c_1}{\alpha+1}\right) - 2\left(\frac{c_1}{\alpha+1}\right)^2$$

and

$$a_3 = \frac{c_2}{2(2\alpha+1)} + \frac{c_1^{\;\;2}}{2\cdot 2^s(2\alpha+1)} + \frac{3c_1^2}{2(\alpha+1)(2\alpha+1)} - \frac{c_1^2}{(\alpha+1)^2(2\alpha+1)} \quad (3.7)$$

$$a_4 = \frac{\alpha c_1^3}{3^s} + \frac{c_1^3}{3.3^s} + \frac{2\alpha c_1^3}{2^s(2\alpha+1)} + \frac{2\alpha c_1^3}{(1+\alpha)^2} + \frac{3\alpha c_1^3}{2^s(\alpha+1)} + \frac{6\alpha c_1^3}{(2\alpha+1)(\alpha+1)} - \frac{3\alpha^2 c_1^3}{2(2\alpha+1)(\alpha+1)^2}$$

$$- \frac{\alpha^2 c_1^3}{2.2^s(2\alpha+1)(\alpha+1)} + \frac{2\alpha c_1 c_2}{2^s} + \frac{2\alpha c_1 c_2}{(2\alpha+1)} + \frac{3\alpha c_1 c_2}{(\alpha+1)} - \frac{\alpha^2 c_1 c_2}{2(2\alpha+1)(\alpha+1)} + \alpha c_3$$

$$+ \frac{2c_1^3}{3.2^s(2\alpha+1)} + \frac{2c_1^3}{3(1+\alpha)^2} + \frac{c_1^3}{2^s(\alpha+1)} + \frac{2c_1^3}{(2\alpha+1)(\alpha+1)} - \frac{15\alpha c_1^3}{(2\alpha+1)(\alpha+1)^2}$$

$$- \frac{11\alpha c_1^3}{3.2^s(2\alpha+1)(\alpha+1)} + \frac{\alpha^2 c_1^3}{(2\alpha+1)(\alpha+1)^3} + \frac{2c_1 c_2}{3.2^s} + \frac{2c_1 c_2}{3(2\alpha+1)} + \frac{c_1 c_2}{\alpha+1}$$

$$- \frac{11\alpha c_1 c_2}{3(2\alpha+1)(\alpha+1)} + \frac{c_1 c_2}{3} - \frac{29 c_1^3}{6(2\alpha+1)(\alpha+1)^2} - \frac{7c_1^3}{6.2^s(2\alpha+1)(\alpha+1)}$$

$$+ \frac{22\alpha c_1^3}{3(2\alpha+1)(\alpha+1)^3} - \frac{7c_1 c_2}{6(2\alpha+1)(\alpha+1)} \frac{7c_1^3}{3(2\alpha+1)(\alpha+1)^3}. \quad (3.8)$$

Using (3.3) in (3.5), (3.7) and (3.8), we have

$$|a_2| \le \frac{1}{\alpha+1}$$

$$|a_3| \le \frac{1}{2.2^s(2\alpha+1)} + \frac{3}{2(2\alpha+1)(\alpha+1)} + \frac{1}{2(2\alpha+1)} - \frac{1}{(2\alpha+1)(\alpha+1)^2}$$

$$|a_4| \le \frac{3\alpha+1}{3.3^s} + \frac{1}{2^s}\left\{\left(\frac{6\alpha+2}{3\alpha+1}\right) + \left(\frac{3\alpha+1}{\alpha+1}\right) + \left(\frac{6\alpha-2}{3(2\alpha+1)}\right) - \left(\frac{3\alpha^2-29}{6(2\alpha+1)(\alpha+1)}\right)\right\}$$

$$+ \frac{1}{3(\alpha+1)^2}\left\{(6\alpha+2) + \left[\frac{6\alpha^2+44\alpha - 7}{2{*}2\alpha+1)(\alpha+1)}\right] - \left[\frac{9\alpha^2+90\alpha+29}{2(2\alpha+1)}\right]\right\}$$

$$+ \frac{1}{3(2\alpha+1)}\left\{(6\alpha^2+9\alpha+2) - \left(\frac{3\alpha^2+14\alpha+5}{\alpha+1}\right)\right\} + \frac{8\alpha+5}{3(\alpha+1)}$$

This completes the proof.

Taking $s = 0,\ \alpha = 0$ and $s = 0, \alpha = 1$, respectively, we obtain the following corollaries

Corollary 3.2 If the function $f(z)$ given by (1.1) be in the class $S(0, \Psi i_0)$.Then

$(\alpha_2) = 1$

$(\alpha_3) = \frac{7}{2}$

$(\alpha_4) = \frac{39}{6}$

Corollary 3.3 If the function $f(z)$ given by (1.1) be in the class $S(1, \Psi i_0)$.Then

$(a_2) = (a_3) \le \frac{1}{2}$

$(a_4) \le \frac{163}{18}$

3.2 Fekete-Szegö inequalities for the function class $S\prec\alpha, \Psi i_s\prec$

Theorem 3.2: If the function $f(z)$ given by (1.1) be in the class $S(\alpha, \Psi i_s)$.Then

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{1}{2(1+2\alpha)}\left(\frac{1}{(\alpha+1)^2} - \frac{1}{2^s} - \frac{3}{\alpha+1}\right) & \mu = 0 \\ \left[\frac{1}{(\alpha+1)^2} - \left(\frac{1}{2^s} + \frac{3}{\alpha+1} - \frac{1}{(\alpha+1)^2}\right)\right] & \mu = 1 \\ \frac{1}{2(1+2\alpha)}\left(\frac{2\mu(1+2\alpha)}{(\alpha+1)^2} - \left(\frac{1}{2^s} + \frac{3}{\alpha+1} - \frac{1}{(\alpha+1)^2}\right)\right) & 0 < \mu < 1 \end{cases}$$

Proof: By Applying (2.3) with (3.5) and (3.7), we have

$$|a_3 - \mu a_2^2| \le \left|\frac{c_2}{2(2\alpha+1)} + \frac{c_1^2}{2\cdot 2^s(2\alpha+1)} + \frac{3c_1^2}{2(\alpha+1)(2\alpha+1)} - \frac{c_1^2}{(2\alpha+1)(1+\alpha)^2} - \frac{\mu c_1^2}{(1+\alpha)^2}\right|$$

$$= \left|\frac{1}{2(2\alpha+1)}\left\{c_2 - \left(\frac{2\mu(1+2\alpha)}{(\alpha+1)^2} - (\frac{1}{2^s} + \frac{3}{\alpha+1} - \frac{1}{(\alpha+1)^2})\right) c_1^2\right\}\right|$$

Then from (3.3), we have

$$|a_3 - \mu a_2^2| \le \frac{1}{2(2\alpha+1)} \max\left\{1, \left|\frac{2\mu(1+2\alpha)}{(\alpha+1)^2} - \left(\frac{1}{2^s} + \frac{3}{\alpha+1} - \frac{1}{(\alpha+1)^2}\right)\right|\right\}.$$

Discussion and Conclusion

In this paper we have made use of the principle of subordination to define a new subclass $S(\alpha, \Psi i_s)$ via modification of Jonquiére's function (popularly known as Polylogarithm function) and for the class defined, we obtained the first few coefficient bounds $|a_2| - |a_4|$ and the non-linear functional $|a_3 - \mu a_2^2|$ in order to establish relevant connection of our work to Fekete-Szegö classical theorem.

The approach used in getting our new results is the techniques used by theearlier authors. For example ((Alhindin and Darus, 2014); (Altinkaya &Yalcin, 2016); (Akgül, 2017); (Akgül, 2018); (Sofo, 2019); (Oladipo, 2019); (Olukoya and Oyekan, 2020) and (Oyekan and Awolere, 2020)). Further areas of fruitful research for the class studied are those related to $|a_n|, n = 5.$ and the non-linear functional $|a_2 a_4 - a_2^3|$. Although, achieving this might be a herculean task but not untrack able.It is also possible to get another set of new results which may be akin to the ones in this paper by another modified form of the function $Li_s(z)$.

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